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A Reduction of Finitely Expandable Deep Pushdown Automata

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Abstract
For a positive integer n, n-expandable deep pushdown automata always contain no more than n occurrences of non-input symbols in their pushdowns during any computation. As its main result, the present paper demonstrates that these automata are as powerful as the same automata with only two non-input pushdown symbols—\$ and #, where # always appears solely as the pushdown bottom as it would in the regular pushdown automaton. The paper demonstrates an infinite hierarchy of language families that follows from this main result. In its conclusion, the paper suggests open problems and topics for the future investigation.

Keywords: Deep Pushdown Automata, Finite Expandability, Reduction, Non-Input Pushdown Symbols

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1. Introduction

In essence, deep pushdown automata represent language-accepting models based upon new stack-like structures, which can be modified deeper than on their top. As a result, these automata can make expansions deeper in their pushdown lists as opposed to ordinary pushdown automata, which can expand only the very pushdown top. At present, the study of deep pushdown automata represent a vivid trend in formal language theory (see [1, 2, 3, 4]). The present paper makes a contribution to this trend.

This paper narrows its attention to n-expandable deep pushdown automata, where n is a positive integer. In essence, during any computation, their pushdown lists contain #, which always appears as the pushdown bottom, and no more than $n-1$ occurrences of other non-input symbols. As its main result, the paper demonstrates how to reduce the number of their non-input pushdown symbols different from # to one symbol, denoted by \$, without affecting the power of these automata. Based on this main result, the paper establishes an infinite hierarchy of language families resulting from these reduced versions of n-expandable deep pushdown automata. More precisely, consider n-expandable deep pushdown automata with pushdown alphabets containing #, \$, and input symbols. The paper shows that $(n+1)$-expandable versions of these automata are stronger than their n-expandable versions, for every positive integer n. In addition, it points out that these automata with # as its only non-input symbol characterize the family of regular languages. In its conclusion, this paper formulates several open problem areas related to the subject of this paper for the future study.

The paper is organized as follows. Section 2 gives all the definitions needed to follow the paper. Section 3 establishes all the results sketched above, and in its conclusion, it also brings the reader's attention to several open problems.

2. Preliminaries and Definitions

We assume that the reader is familiar with formal language theory (see Harrison [5] or Meduna [6, 7]). Let $\mathbb{N}$ denote the set of all positive integers. For an alphabet Γ, Γ^* represents the free monoid generated by Γ under the operation of concatenation. The identity of Γ^* is denoted by ε. For $w \in \Gamma^*$, $|w|$ denotes the length of w.

A *deep pushdown automaton* (Deep PDA) is a 7-tuple $M = (Q, \Sigma, \Gamma, R, s, S, F)$, where Q is a finite set of states, Σ is a input alphabet, Γ is a pushdown alphabet, $\Sigma \subseteq \Gamma$ is an input alphabet, $s \in Q$ is the start state, $S \in \Gamma \setminus \Sigma$ is the start pushdown symbol, and $F \subseteq Q$ is the set of final states. $\Gamma \setminus \Sigma$ contains the bottom pushdown symbol denoted by #. In what follows, $N = \Gamma \setminus (\Sigma \cup \{\#\})$. R is a finite subset of $(\mathbb{N} \times Q \times N \times Q \times (\Gamma \setminus \{\#\})^+) \cup (\mathbb{N} \times Q \times \{\#\} \times Q \times (\Gamma \setminus \{\#\})^* \{\#\})$. R is called the *set of rules*; instead of $(m, q, A, t, v) \in R$, we write $mqA \rightarrow tv$ throughout.

A *configuration* of M is any member of $Q \times \Sigma^* \times (\Gamma \setminus \{\#\})^* \{\#\}$. Let Ξ denote the set of all configurations of M. Next, we define three binary relations over Ξ — ${}_p\vdash$, ${}_e\vdash$, and $\vdash$. Let $q, t \in Q$, $x \in \Sigma^*$, $z \in (\Gamma \setminus \{\#\})^* \{\#\})$.

1. Let $a \in \Sigma$; then, $(q, ax, az)\ {}_p\vdash (q, x, z)$.
2. Let $mqA \rightarrow tv \in R$, $z = uAw$, $u \in (\Gamma \setminus \{\#\})^*$, u contains $m-1$ occurrences of symbols from N, either $A \in N$, $v \in (\Gamma \setminus \{\#\})^+$ and $w \in (\Gamma \setminus \{\#\})^* \{\#\}$ or $A = \#$, $v \in (\Gamma \setminus \{\#\})^* \{\#\}$, and $w = \varepsilon$; then, $(q, x, uAw)\ {}_e\vdash (t, x, uvw)$.
3. Let $\alpha, \beta \in \Xi$; $\alpha \vdash \beta$ if and only if $\alpha\ {}_p\vdash \beta$ or $\alpha\ {}_e\vdash \beta$.

Intuitively, in ${}_p\vdash$ and ${}_e\vdash$, p and e stand for *pop* and *expansion*, respectively. Consider 2 above; to express that $(q, x, uAw)\ {}_e\vdash (q, x, uvw)$ is made according to $mqA \rightarrow tv$, write $(q, x, uAw)\ {}_e\vdash (t, x, uvw)\ [mqA \rightarrow tv]$. If $\alpha, \beta \in \Xi$, $\alpha \vdash \beta$ in M, we say that M makes a *move* from α to β.

In the standard manner, extend ${}_e\vdash$, ${}_p\vdash$, and $\vdash$ to ${}_e\vdash^i$, ${}_p\vdash^i$, and $\vdash^i$, respectively, for $i \geq 0$; then, based on ${}_e\vdash^i$, ${}_p\vdash^i$, and $\vdash^i$, define ${}_e\vdash^+$, ${}_e\vdash^*$, ${}_p\vdash^+$, ${}_p\vdash^*$, $\vdash^+$, and $\vdash^*$. The language of M, $L(M)$, is defined as $L(M) = \{w \mid (s, w, S\#) \vdash^* (f, \varepsilon, \#) \text{ in } M, w \in \Sigma^*, f \in F\}$.

Let $n \in \mathbb{N}$. If during any $\alpha \vdash^* \beta$ in M, $\alpha, \beta \in \Xi$, M has no more than n occurrences of symbols form $\Gamma \setminus \Sigma$ in its pushdown, then M is an *n-expandable Deep PDA*.

A *right-linear grammar* is a quadruple $G = (N, T, P, S)$, where N is an alphabet of nonterminals, T is an alphabet of terminals such that $N \cap T = \emptyset$, P is a finite subset of $N \times T^*(N \cup \{\varepsilon\})$, and $S \in N$. P is called the *set of rules* in G; instead of $(A, x) \in P$, we write $A \rightarrow x$. Define the language of G, $L(G)$, as usual (see [6]).

Let $n, r \in \mathbb{N}$, ${}_n$**DPDA** denotes the language family accepted by n-expandable Deep PDA. ${}_n$**DPDA**$_r$ denotes the language family accepted by n-expandable deep pushdown automata with # and no more than $(r-1)$ non-input pushdown symbols. **Reg** denotes the regular language family. Recall that **Reg** is characterized by right-linear grammars (see Theorem 7.2.2. in [6]).

3. Result

Next, we establish Lemma 3.1, which implies the main result of this paper.

Lemma 3.1. *Let $n \in \mathbb{N}$. For every n-expandable Deep PDA M, there exists an n-expandable Deep PDA M_R such that $L(M) = L(M_R)$ and M_R contains only two non-input pushdown symbols—\$ and #.*

Proof. *Construction.* Let $n \in \mathbb{N}$. Let

$$M = (Q, \Sigma, \Gamma, R, s, S, F)$$

be an n-expandable Deep PDA. Recall that rules in R are of the form $mqA \rightarrow tv$, where $m \in \mathbb{N}$, $q, t \in Q$, either $A \in N$ and $v \in (\Gamma \setminus \{\#\})^+$ or $A = \#$ and $v \in (\Gamma \setminus \{\#\})^* \{\#\}$, where # denotes the pushdown bottom.

Let \$ be a new symbol, $\$ \notin Q \cup \Gamma$, and let homomorphisms f and g over Γ^* be defined as $f(A) = A$ and $g(A) = \$$, for every $A \in N$, and $f(a) = \varepsilon$ and $g(a) = a$, for every $a \in (\Sigma \cup \{\#\})$. Next, we construct an n-expandable Deep PDA

$$M_R = (Q_R, \Sigma, \Sigma \cup \{\$, \#\}, R_R, s_R, \$, F_R)$$

by performing 1 through 4, given next:

1. Add $m\langle q; uAz\rangle\$ \rightarrow \langle t; uf(v)z\rangle g(v)$ to R_R and add $\langle q; uAz\rangle$, $\langle t; uf(v)z\rangle$ to Q_R if $mqA \rightarrow tv \in R$, $u, z \in N^*$, $|u| = m-1$, $|z| \leq n-m-1$, $|uf(v)z| < n$, $m \in \mathbb{N}$, $q, t \in Q$, $A \in N$, and $v \in (\Gamma \setminus \{\#\})^+$;
2. Add $m\langle q; u\rangle\# \rightarrow \langle t; uf(v)\rangle g(v)\#$ to R_R and add $\langle q; u\rangle$, $\langle t; uf(v)\rangle$ to Q_R if $mq\# \rightarrow tv\# \in R$, $u \in N^*$, $|u| = m-1$, $|uf(v)| < n$, $m \in \mathbb{N}$, $q, t \in Q$, and $v \in (\Gamma \setminus \{\#\})^*$;
3. Set $s_R = \langle s; S\rangle$;
4. Add all $\langle t; u\rangle$ to F_R, where $t \in F$, $u \in N^*$, $u < n$.

Later in this proof, we demonstrate that $L(M) = L(M_R)$.

Basic Idea States in Q_R include not only the states corresponding to the states in Q but also strings of non-input symbols. Whenever M pushes a non-input symbol onto the pushdown, M_R records this information within its current state and pushes \$ onto the pushdown instead.

By Lemma 3.1. in [8], any n-expandable Deep PDA M can accept every $w \in L(M)$ so all expansions precede all pops during the accepting process. Without any loss of generality, we assume that M and M_R work in this way in what follows, too.

To establish $L(M) = L(M_R)$, we prove the following four claims.

Claim 3.1. *Let* $(s, w, S\#) \vdash^j (q, v, x\#)$ *in* M*, where* $s, q \in Q$*,* $w, v \in \Sigma^*$*, and* $x \in (\Gamma \setminus \{\#\})^*$*. Then,* $(\langle s; S\rangle, w, \$\#) \vdash^* (\langle q; f(x)\rangle, v, g(x)\#)$ *in* M_R*, where* $\langle s; S\rangle$*,* $\langle q; f(x)\rangle \in Q_R$*, and* $g(x) \in (\Sigma \cup \{\$\})^*$*.*

This claim is proved by induction on $j \geq 0$.

Basis Let $j = 0$, so $(s, w, S\#) \vdash^0 (s, w, S\#)$ in M, where $s \in Q$ and $S \in N$. Then, from 3 in the construction, we obtain

$$(\langle s; S\rangle, w, \$\#) \vdash^0 (\langle s; S\rangle, w, \$\#)$$

in M_R, so the basis holds.

Induction Hypothesis Assume there is $i \geq 0$ such that Claim 3.1 holds true for all $0 \leq j \leq i$.

Induction Step Let $(s, w, S\#) \vdash^{i+1} (q, w, x\#)$ in M, where $x \in (\Gamma \setminus \{\#\})^*$, $s, q \in Q$, $w \in \Sigma^*$, $k, \ell \geq 1$, $k + \ell < n$.

Since $i + 1 \geq 1$, we can express $(s, w, S\#) \vdash^{i+1} (q, w, x\#)$ as

$$(s, w, S\#) \vdash^i (t, w, x_0 A_1 x_1 \ldots A_m \ldots A_k x_k \#)$$

$$\vdash (q, w, x_0 A_1 x_1 \ldots A_{m-1} x_{m-1} y_0 B_1 y_1 \ldots B_\ell y_\ell \; x_m A_{m+1} \ldots x_{k-1} A_k x_k \#)$$

$$[mtA_m \rightarrow qy_0 B_1 y_1 \ldots B_\ell y_\ell]$$

where $A_1, \ldots, A_k, B_1, \ldots, B_\ell \in N$ and $x_0 x_1 \ldots x_k, y_0 y_1 \ldots y_\ell \in \Sigma^*$, $t \in Q$. By the induction hypothesis, we have

$$(\langle s; S\rangle, w, \$\#) \vdash^* (\langle t; A_1 \ldots A_m \ldots A_k\rangle, w, x_0 \$ x_1 \$ \ldots \$ x_k \#)$$

Since $mtA_m \rightarrow qy_0 B_1 y_1 \ldots B_\ell y_\ell \in R$, we also have

$$m\langle t; A_1 \ldots A_m \ldots A_k\rangle \$ \rightarrow \langle q; A_1 \ldots A_{m-1} B_1 \ldots B_\ell A_{m+1} \ldots A_k\rangle y_0 \$ y_1 \$ \ldots \$ y_\ell \in R_R$$

(see 1 in the construction). Thus,

$$(\langle t; A_1 \ldots A_m \ldots A_k\rangle, w, x_0 \$ x_1 \$ \ldots \$ x_k \#)$$

$$\vdash (\langle q; A_1 \ldots A_{m-1} B_1 \ldots B_\ell A_{m+1} \ldots A_k\rangle, w, x_0 \$ x_1 \$ \ldots \$ x_{m-1} y_0 \$ y_1 \$ \ldots \$ y_\ell x_m \$ \ldots \$ x_k \#)$$

$$[m\langle t; A_1 \ldots A_m \ldots A_k\rangle \$ \rightarrow \langle q; A_1 \ldots A_{m-1} B_1 \ldots B_\ell A_{m+1} \ldots A_k\rangle y_0 \$ y_1 \$ \ldots \$ y_\ell]$$

Analogically, we can prove the induction step for the case when # is rewritten (see 2 in the construction). Therefore, Claim 3.1 holds true.

Claim 3.2. $L(M) \subseteq L(M_R)$.

Proof. Consider Claim 3.1 for $v = \varepsilon$, $q \in F$, and $x = \varepsilon$. Under this consideration Claim 3.1 implies Claim 3.2. □

Claim 3.3. *Let* $(\langle s; S\rangle, w, \$\#) \vdash^j (\langle q; A_1 \ldots A_k\rangle, v, x\#)$ *in* M_R*, where* $s_R = \langle s; S\rangle$*,* $\langle q; A_1 \ldots A_k\rangle \in Q_R$*,* $w, v \in \Sigma^*$*,* $A_1, \ldots, A_k \in N$*,* $x = x_0 \$ x_1 \$ \ldots \$ x_k$*, and* $x_0 \ldots x_k \in \Sigma^*$*. Then,* $(s, w, S\#) \vdash^* (q, v, x_0 A_1 x_1 \ldots A_k x_k \#)$ *in* M*, where* $s, q \in Q$*.*

Proof. This claim is proved by induction on $j \geq 0$.

Basis. Let $j = 0$, so $(\langle s; S\rangle, w, \$\#) \vdash^0 (\langle s; S\rangle, w, \$\#)$ in M_R, where $s_R = \langle s; S\rangle$. From 3 in the construction, we have

$$(s, w, S\#) \vdash^0 (s, w, S\#)$$

in M, so the basis holds.

Induction Hypothesis. Assume there is $i \geq 0$ such that Claim 3.3 holds true for $0 \leq j \leq i$.

Induction Step. Let $(\langle s; S\rangle, w, \$\#) \vdash^{i+1} (\langle q; A_1 \ldots A_k\rangle$, w, $x_0 \$ x_1 \$ \ldots \$ x_k \#)$ in M_R, where $\langle s; S\rangle, \langle q; A_1 \ldots A_k\rangle \in Q_R$, $A_1, \ldots, A_k \in N$, $w \in \Sigma^*$, and $x_0 \ldots x_k \in \Sigma^*$, $k, \ell \geq 1$, $k + \ell < n$. Since $i + 1 \geq 1$, we can express $(\langle s; S\rangle$, w, $\$\#) \vdash^{i+1} (\langle q; A_1 \ldots A_k\rangle$, w, $x_0 \$ x_1 \$ \ldots \$ x_k \#)$ as

$$(\langle q; (\langle s; S\rangle, w, \$\#) \vdash^i (\langle t; A_1 \ldots A_m \ldots A_k\rangle, w, x_0 \$ x_1 \$ \ldots \$ x_k \#)$$

$$\vdash A_1 \ldots A_{m-1} B_1 \ldots B_\ell A_{m+1} \ldots A_k\rangle, w, x_0 \$ x_1 \$ \ldots \$ x_{m-1} y_0 \$ y_1 \$ \ldots \$ y_\ell x_m \$ \ldots \$ x_k \#)$$

$$[m\langle t; A_1 \ldots A_m \ldots A_k\rangle \$ \rightarrow \langle q; A_1 \ldots A_{m-1} B_1 \ldots B_\ell A_{m+1} \ldots A_k\rangle y_0 \$ y_1 \$ \ldots \$ y_\ell]$$

By the induction hypothesis, we obtain

$$(s, w, S\#) \vdash^i (t, w, x_0 A_1 x_1 \ldots A_m \ldots A_k x_k \#)$$

Since $m\langle t; A_1 \ldots A_m \ldots A_k\rangle \$ \rightarrow \langle q; A_1 \ldots A_{m-1} B_1 \ldots B_\ell A_{m+1} \ldots A_k\rangle y_0 \$ y_1 \$ \ldots \$ y_\ell \in R_R$, we also have $mtA_m \rightarrow qy_0 B_1 y_1 \ldots B_\ell y_\ell \in R$ as follows from 1 in the construction. We obtain

$$(t, w, x_0 A_1 x_1 \ldots A_m \ldots A_k x_k \#)$$

$$\vdash (q, w, x_0 A_1 x_1 \ldots A_{m-1} x_{m-1} y_0 B_1 y_1 \ldots B_\ell y_\ell x_m A_{m+1} \ldots x_{k-1} A_k x_k \#)$$

$$[mtA_m \rightarrow qy_0 B_1 y_1 \ldots B_\ell y_\ell]$$

Analogically, we can prove the case when # is expanded (see 2 in the construction). Therefore, Claim 3.3 holds true. □

Claim 3.4. $L(M_R) \subseteq L(M)$.

Proof. Consider Claim 3.3 with $v = \varepsilon$, $\langle q; A_1 \ldots A_k\rangle \in F_R$, and $x = \varepsilon$. Under this consideration, Claim 3.3 implies Claim 3.4. □

As $L(M) \subseteq L(M_R)$ (see Claim 3.2) and $L(M_R) \subseteq L(M)$ (see Claim 3.4), $L(M_R) = L(M)$. Thus, Lemma 3.1 holds. $\square$

Example Take this three-expandable Deep PDA $M = (\{s,q,t\},\{a,b,c\},\{a,b,c,A,S,\#\}, R,s,S,\{f\})$ where

$$\begin{aligned} R = \{&1sS \to qAA, \\ &1qA \to fab, \\ &1fA \to fc, \\ &1qA \to taAb, \\ &2tA \to qAc\} \end{aligned}$$

By the construction given in the proof of Lemma 3.1, we construct $M_R = (Q_R, \{a,b,c\}, \{a,b,c,\$,\#\}, R_R, \langle s;S\rangle, \$, \{\langle f;A\rangle, \langle f;\varepsilon\rangle\})$, where $Q_R = \{\langle s;S\rangle, \langle q;AA\rangle, \langle f;A\rangle, \langle f;\varepsilon\rangle, \langle t;AA\rangle\}$ and

$$\begin{aligned} R_R = \{&1 : 1\langle s;S\rangle\$ \to \langle q;AA\rangle\$\$, \\ &2 : 1\langle q;AA\rangle\$ \to \langle f;A\rangle ab, \\ &3 : 1\langle f;A\rangle\$ \to \langle f;\varepsilon\rangle c, \\ &4 : 1\langle q;AA\rangle\$ \to \langle t;AA\rangle a\$b, \\ &5 : 2\langle t;AA\rangle\$ \to \langle q;AA\rangle\$c, \\ &6 : 1\langle s;SA\rangle\$ \to \langle q;AAA\rangle\$\$, \\ &7 : 1\langle s;SS\rangle\$ \to \langle q;AAS\rangle\$\$, \\ &8 : 1\langle q;AA\rangle\$ \to \langle f;A\rangle ab, \\ &9 : 1\langle q;AS\rangle\$ \to \langle f;S\rangle ab, \\ &10 : 1\langle f;AA\rangle\$ \to \langle f;A\rangle c, \\ &11 : 1\langle f;AS\rangle\$ \to \langle f;S\rangle c, \\ &12 : 1\langle q;A\rangle\$ \to \langle t;A\rangle a\$b, \\ &13 : 1\langle q;AS\rangle\$ \to \langle t;AS\rangle a\$b, \\ &14 : 2\langle t;SA\rangle\$ \to \langle q;SA\rangle\$c\} \end{aligned}$$

For instance, M_R makes

$$\begin{aligned} (\langle s;S\rangle, aabbcc, \$\#) \ &{}_e\vdash (\langle q;AA\rangle, aabbcc, \$\$\#) && (1) \\ &{}_e\vdash (\langle t;AA\rangle, aabbcc, a\$b\$\#) && (4) \\ &{}_p\vdash (\langle t;AA\rangle, abbcc, \$b\$\#) \\ &{}_e\vdash (\langle q;AA\rangle, abbcc, \$b\$c\#) && (5) \\ &{}_e\vdash (\langle f;A\rangle, abbcc, abb\$c\#) && (2) \\ &{}_p\vdash (\langle f;A\rangle, cc, \$c\#) \\ &{}_e\vdash (\langle f;\varepsilon\rangle, cc, cc\#) && (3) \\ &{}_p\vdash (\langle f;\varepsilon\rangle, \varepsilon, \#) \end{aligned}$$

Theorem 3.1. *For all $n \geq 1$, ${}_n$**DPDA** = ${}_n$**DPDA**$_2$.*

Proof. This theorem follows from Lemma 3.1. $\square$

Corollary 3.1. *For all $n \geq 1$, ${}_n$**DPDA**$_2 \subset {}_{n+1}$**DPDA**$_2$.*

Proof. This corollary follows from Theorem 3.1 in this paper and Corollary 3.1. in [8]. $\square$

Can we reformulate Theorem 3.1 and Corollary 3.1 in terms of ${}_n$**DPDA**$_1$? The answer is no as we show next.

Lemma 3.2. *Let $M = (Q,\Sigma,\Gamma,R,s,S,F)$ be a Deep PDA with $\Gamma \setminus \Sigma = \{\#\}$. Then, there is a right-linear grammar G such that $L(G) = L(M)$.*

Proof. Let $M = (Q,\Sigma,\Gamma,R,s,S,F)$ with $\Gamma \setminus \Sigma = \{\#\}$. Thus, every rule in R is of the form $1q\# \to px\#$, where $q,p \in Q$, $x \in \Sigma^*$. Next, we construct a right-linear grammar $G = (Q,\Sigma,P,s)$ so $L(M) = L(G)$. We construct P as follows:

1. For every $1q\# \to px\# \in R$, where $p,q \in Q$, $x \in \Sigma^*$, add $q \to xp$ to P;
2. For every $f \in F$, add $f \to \varepsilon$ to P.

A rigorous proof that $L(M) = L(G)$ is left to the reader. $\square$

Theorem 3.2. **Reg** = ${}_1$**DPDA**$_1$ = ${}_n$**DPDA**$_1$, *for any $n \geq 1$.*

Proof. Let $n \geq 1$. **Reg** $\subseteq {}_1$**DPDA**$_1$ = ${}_n$**DPDA**$_1$ is clear. Recall that right-linear grammars characterize **Reg**, so ${}_n$**DPDA**$_1 \subseteq$ **Reg** follows from Lemma 3.2. Thus, **Reg** = ${}_n$**DPDA**$_1$. $\square$

Corollary 3.2. **Reg** = ${}_1$**DPDA**$_1$ = ${}_n$**DPDA**$_1 \subset {}_n$**DPDA**$_2$, *for all $n \geq 2$.*

Proof. Let $n \geq 1$. As obvious, ${}_1$**DPDA**$_1$ = ${}_n$**DPDA**$_1 \subseteq {}_n$**DPDA**$_2$. Observe that

$$\{a^n b^n \mid n \geq 1\} \in {}_n\mathbf{DPDA}_2 \setminus {}_n\mathbf{DPDA}_1$$

Therefore, Corollary 3.2 holds. $\square$

4. Conclusions

In the present paper, we have reduced finitely expandable Deep PDAs with respect to the number of non-input pushdown symbols.

Before closing this paper, we suggest some open problem areas related to this subject for the future investigation.

1. Can we reduce these automata with respect to the number of states?
2. Can we simultaneously reduce them with respect to the number of both states and non-input pushdown symbols?
3. Can we achieve the reductions described above in terms of general Deep PDAs, which are not finitely expandable?

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